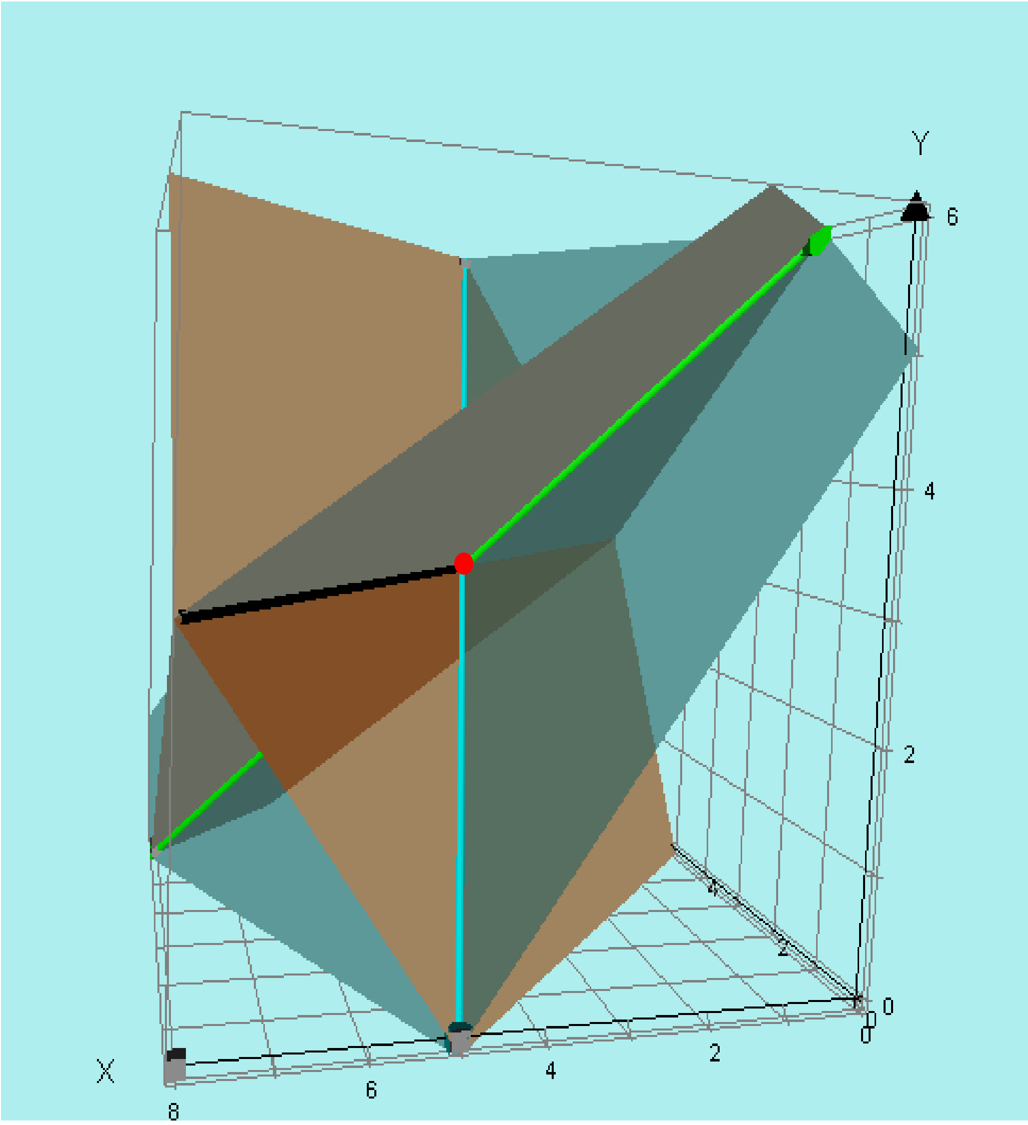
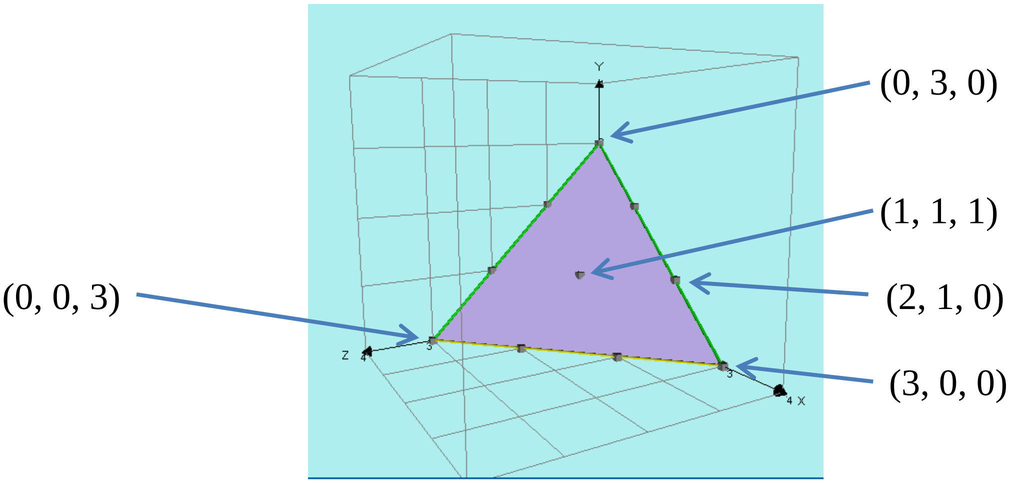
Applying systems of simultaneous equations in solving problems.

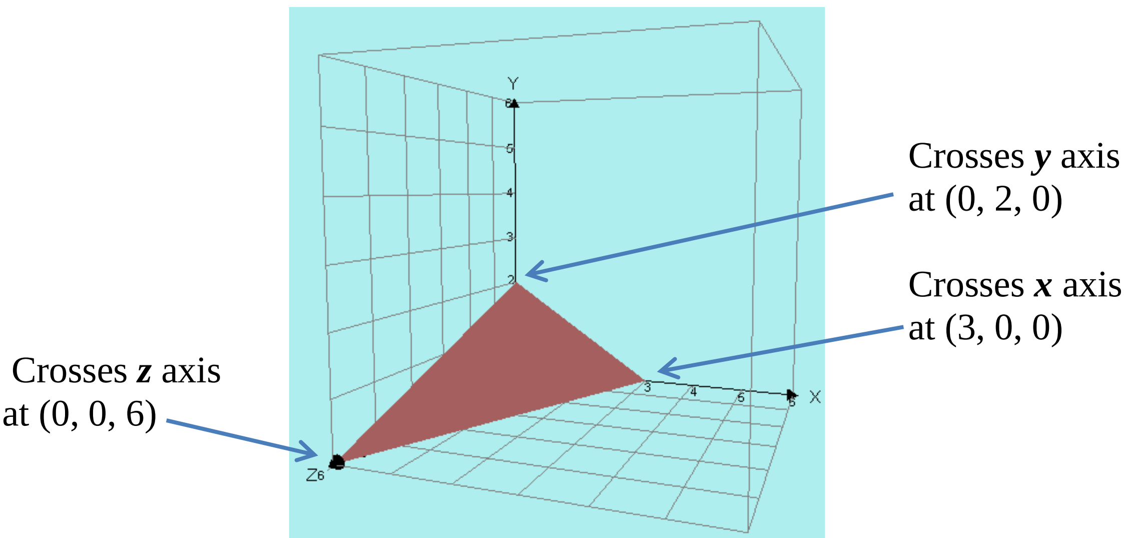


INTERSECTIONS OF PLANES in 3 DIMENSIONS.

1. We are all familiar with equations like $y = 2x + 3$ or in general $y = mx + c$.
   We know that such equations represent **LINES** in the ***x, y*** plane.
   In this case, the gradient is 2 and the y intercept is 3 and we can easily **visualise** the position of any such line.
   Line equations are sometimes written in the form $ax + by = c$ in which case the above line would become $2x - y = -3$. This of course is the same line with gradient 2 and y intercept 3.
   However, equations such as $x + y + z = 3$ do NOT represent **LINES** in 3 dimensional space. They actually represent **PLANES**.
   If we choose some points (x, y, z) which fit this equation we obviously get the following: (3, 0, 0), (0, 3, 0), (0, 0, 3), (1, 1, 1), (1, 2, 0), (1, 0, 2) etc
   If we were to plot these we would get the following plane:



2. It is quite difficult to imagine the position of a plane such as $2x + 3y + z = 6$
   A simple way to do this is to **find where the plane would cross each axis.**
   If we put $x = 0$ *and* $y = 0$ *then* $z = 6$ so it crosses the z axis at (0, 0, 6)
   If we put $y = 0$ *and* $z = 0$ *then* $2x = 6$ *so* $x = 3$ crossing the x axis at (3, 0, 0)
   If we put $x = 0$ *and* $z = 0$ *then* $3y = 6$ *so* $y = 2$ crossing the y axis at (0, 2, 0)

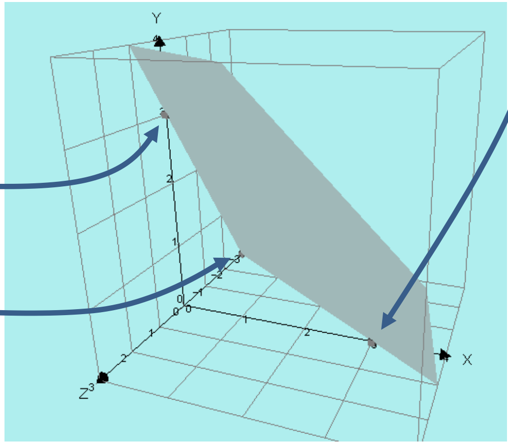


3. If we consider the equation $\boldsymbol{x + y - z = 3}$

Putting ***y and z = 0 we get x = 3*** so it crosses ***x*** axis at (3, 0 ,0)

Putting ***x and z = 0 we get y = 3*** so it crosses ***y*** axis at (0, 3, 0)

But putting ***x and y = 0 we get z = -3*** so it crosses ***z*** axis at (0, 0, -3)



This is obviously far more difficult to visualise since we need to extend the *z* axis to -3.

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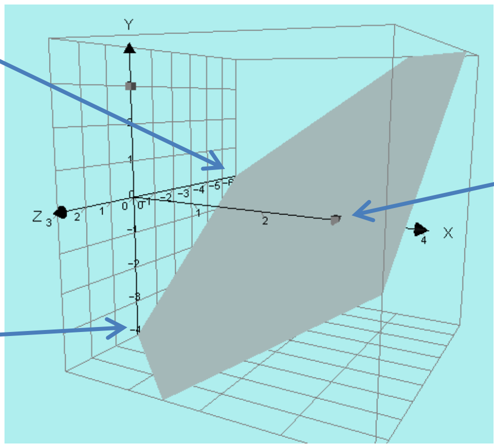
4. The plane $\boldsymbol{4x - 3y - 2z = 12}$ crosses the axes at (3, 0, 0), (0, -4, 0), (0, 0, -6)

z axis at (0, 0, -6)



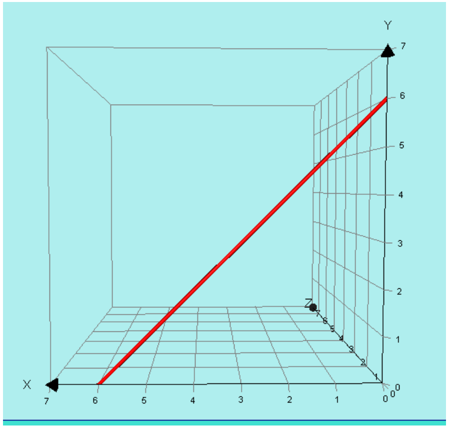
x axis at (3, 0, 0)

y axis at (0, -4, 0)

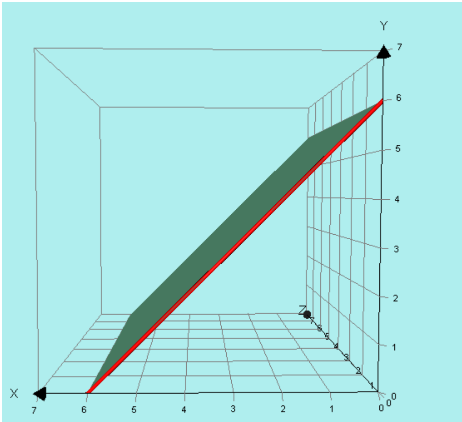
Most of the time, it is not helpful to worry about trying to visualise the position of a plane from its equation.

The AUTOGRAPH program is available to overcome this difficulty.

5. Naturally, an equation with ***x*, *y*** and ***z*** such as $x + y + z = 6$ involves coordinates such as (3, 1, 2) and clearly represents a **PLANE** in 3-D space but an equation such as $x + y = 6$ does not mention ***z*** and although it normally just represents a **LINE** in the ***x*, *y*** plane, it can also represent a **PLANE** in the ***x*, *y*, *z*** coordinate system.
If $x + y = 6$ then ***x*** and ***y*** values are connected but ***z*** can have ANY value.



Above $x + y = 6$ is just a **LINE** in the 2-D ***x*, *y*** plane.

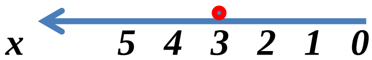


Above $x + y = 6$ is a **PLANE** in 3-D ***x*, *y*, *z*** space.

6. Similarly the equation $x = 3$ **<u>COULD</u>** represent:

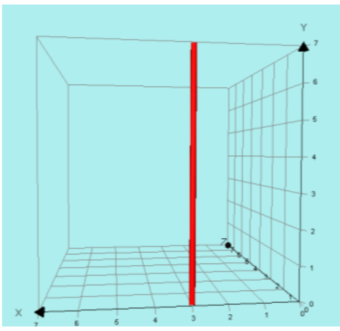
(i) a **<u>single point</u>** on an ***x*** number line.

$x = 3$



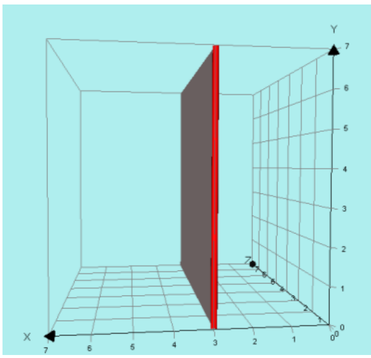
(ii) a **<u>vertical line</u>** in the ***x*, *y*** plane.

$x = 3$



(iii) a **<u>vertical plane</u>** in ***x*, *y*, *z*** space.

$x = 3$



<u>Technique for finding the intersection point of 3 planes.</u>

It is important to be **<u>very precise in labelling equations</u>** in order to describe each step in the process or it could be near impossible for someone else to follow.

1. Consider the 3 planes given by these equations:

$$x + 2y + z = 14 \quad ①$$
$$2x + 2y - z = 10 \quad ②$$
$$x - y + z = 5 \quad ③$$

Adding equations ① and ② we get $3x + 4y = 24$ ④

Adding equations ② and ③ we get $3x + y = 15$ ⑤

Subtracting ④ and ⑤ we get $3y = 9$

$y = 3$

subs in ⑤ so $3x + 3 = 15$

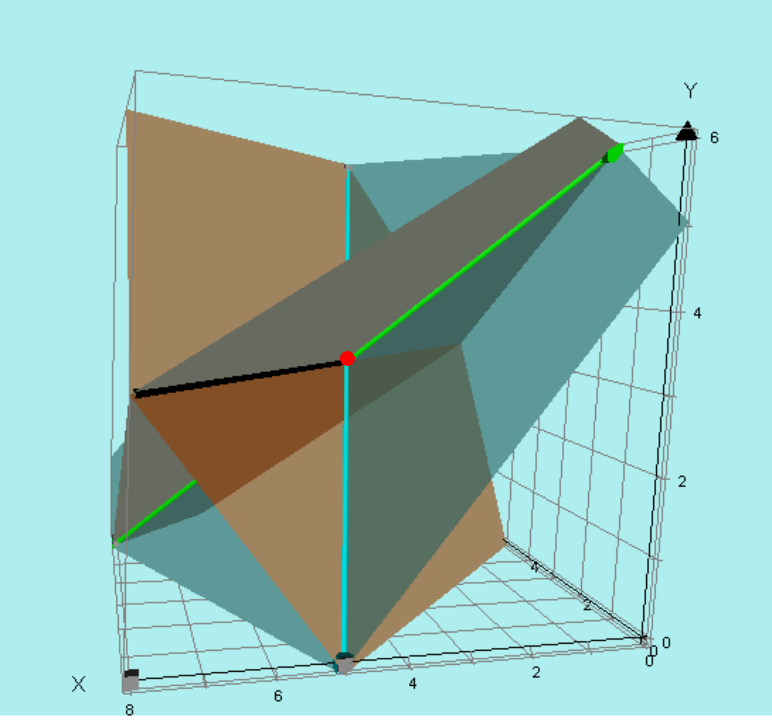
$3x = 12$

$x = 4$

and subs in ③ we get $z = 4$

The intersection point is (4, 3, 4)

The intersection point can be seen from this 3-D graph:



SPECIAL POINT.

In the above problem we added equations ① and ②

$$x + 2y + z = 14 \qquad ①$$
$$2x + 2y - z = 10 \qquad ②$$

…and obtained the equation $3x + 4y = 24$

which can be written as $y = -\frac{3}{4}x + 6$

It is a common misconception that this equation represents the LINE of intersection of the planes. IT DOES NOT.

The actual line of intersection could be written parametrically as:

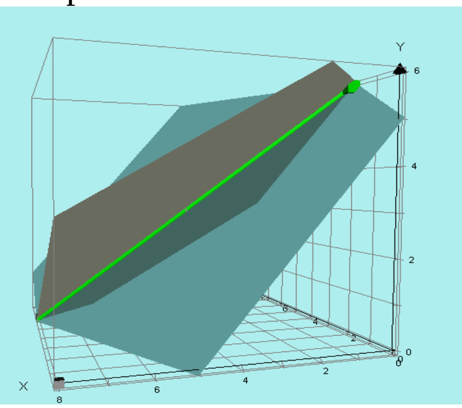
$x = 8 - 8t$, $y = 6t$, $z = 6 - 4t$ which is beyond the scope of this course.

We will NOT use EQUATIONS of LINES in 3-Dimensional space at all.

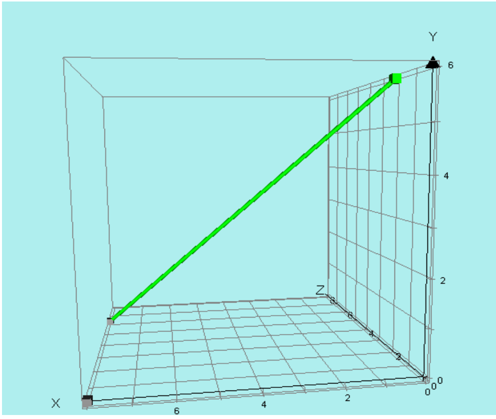
It is better to think of $y = -\frac{3}{4}x + 6$ *as a LINE on the front x, y plane,*

with a gradient of $-\frac{3}{4}$ and y intercept of 6.

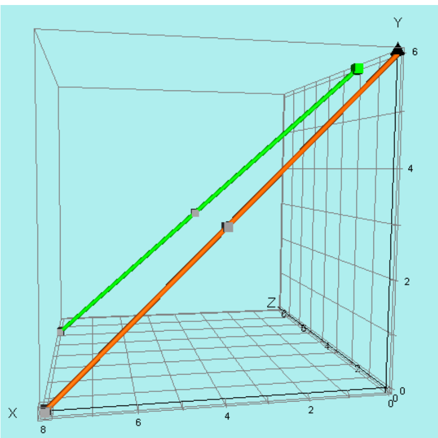
(i) This diagram shows the **green intersection line** of the two planes:



(ii) Taking away the planes and just leaving the **green intersection LINE**:

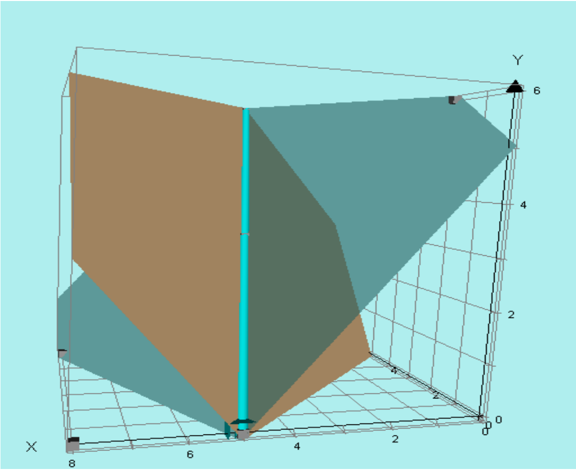


(iii) If we now draw $y = -\frac{3}{4}x + 6$ on the front ***x, y*** plane (**orange** line)



we see that it is the **projection** (or shadow) of the actual line of intersection of the planes, onto the ***x, y*** plane.

(iv) Similarly the **TURQUOISE** line is the intersection of planes ② and ③



(v) …and when we added equations ② and ③ we got the equation ⑤

3x + y = 15 or y = –3x + 15

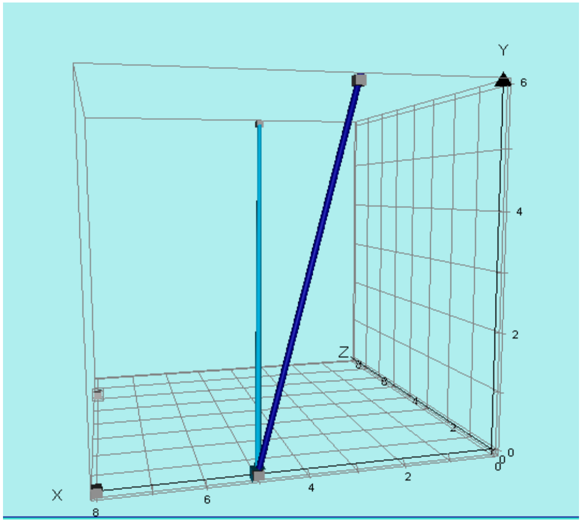
Again this is **not** the line of intersection of planes ② and ③

As before, we could say that

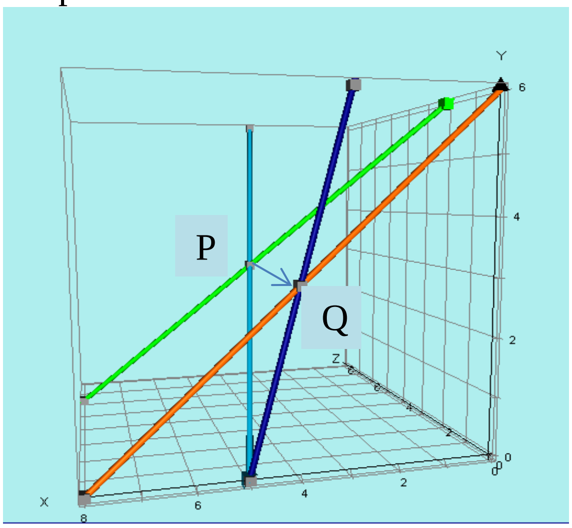
y = – 3x + 15

is just a line in the ***x, y*** plane with a gradient ***– 3*** and ***y*** intercept of ***15.***

(vi) If we now draw this **BLUE** line ***y = -3x + 15*** on the ***x, y*** plane, we can see that it is the projection of the **TURQUOISE** line on the ***x, y*** plane.



(vii) Drawing both projections of the actual intersection lines of the planes, we see that Q is also the projection of P which is the point of intersection of the planes.



This means that, instead of using the **actual lines** of intersection of the planes, we used the **two projected lines of intersection** on the **x, y plane** to find the ***x*** and ***y*** coordinates of the intersection of the three planes. ***x = 4 and y = 3***

Finally we substituted these values into one of the plane equations to find the *z* value. ***x = 4, y = 3 and z = 4***

2. Find the intersection point of the 3 planes given by these equations:

$x + y + z = 8$ ①

$x + y - z = 4$ ②

$2x - y + z = 2$ ③

Adding equations ① and ② we get $2x + 2y = 12$ ④

Subtracting equation ① from ③ we get $x - 2y = -6$ ⑤

Adding ④ and ⑤ we get $3x = 6$

so $x = 2$

Sub in ④ $4 + 2y = 12$

$2y = 8$

so $y = 4$

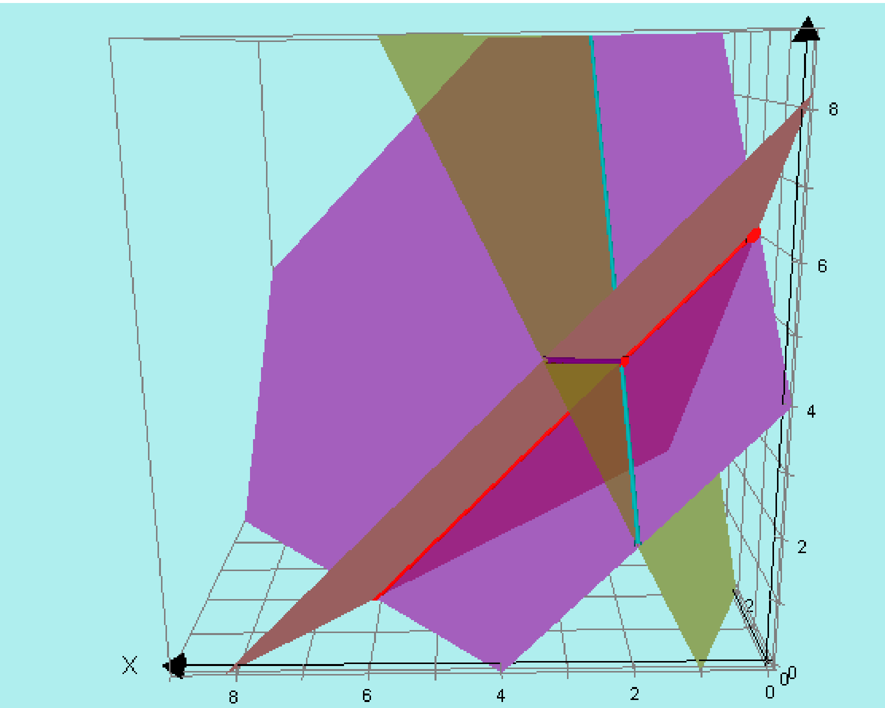
and subs in ① we get:

$2 + 4 + z = 8$

so $z = 2$

The intersection point is (2, 4, 2)

The intersection point can be seen from this 3-D graph:



3. In solving the following 3 equations we need to be **extremely careful** in **labelling equations and detailing the method** because of added difficulties.

$x + y + z = 7$ ①
$4x + 3y + 2z = 23$ ②
$2x - 6y - 3z = -4$ ③

We will choose to eliminate z from ① and ②:

①×2 : $2x + 2y + 2z = 14$ ④
$4x + 3y + 2z = 23$ ②
② – ④: $2x + y = 9$ ⑤

Now we will choose to eliminate z from ① and ③

①×3 : $3x + 3y + 3z = 21$ ⑥
$2x - 6y - 3z = -4$ ③
⑥ + ③: $5x - 3y = 17$ ⑦

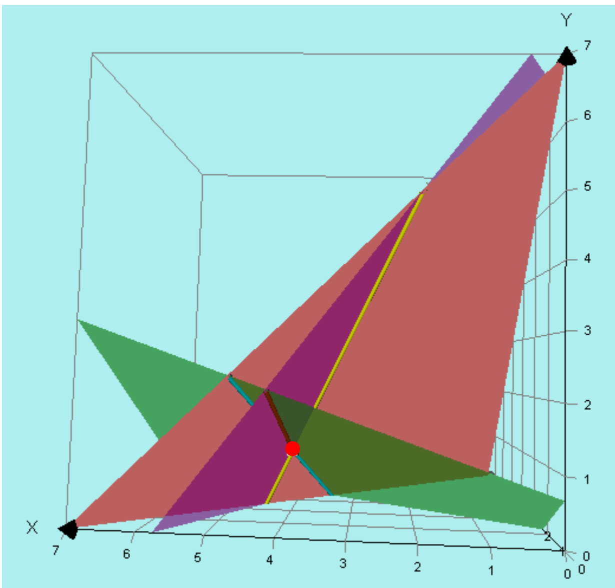
Now we will eliminate y from ⑤ and ⑦

⑤×3 : $6x + 3y = 27$ ⑧
$5x - 3y = 17$ ⑦
⑧ + ⑦: $11x = 44$

$x = 4$
sub in ⑤: $y = 1$
sub in ①: $z = 2$

intersection point is (4, 1, 2)

Note: This can still look very confusing even when the planes are drawn in the Autograph program.



4. Sometimes we have three equations to solve when at least one of them only has two variables.

$3x + 2y + z = 13$ ①

$2x + 5y - 2z = 9$ ②

$4x + 3y = 13$ ③

This situation makes the problem easier because we only have to eliminate ***z*** **once** instead of **twice**.

Eliminating ***z*** from *① and ②:*

①×2 : $6x + 4y + 2z = 26$ ④

$2x + 5y - 2z = 9$ ②

④ + ② : $8x + 9y = 35$ ⑤

③ ×3: $12x + 9y = 39$ ⑥

⑥ - ⑤: $4x = 4$

So $x = 1$

Sub in ③: $4 + 3y = 13$

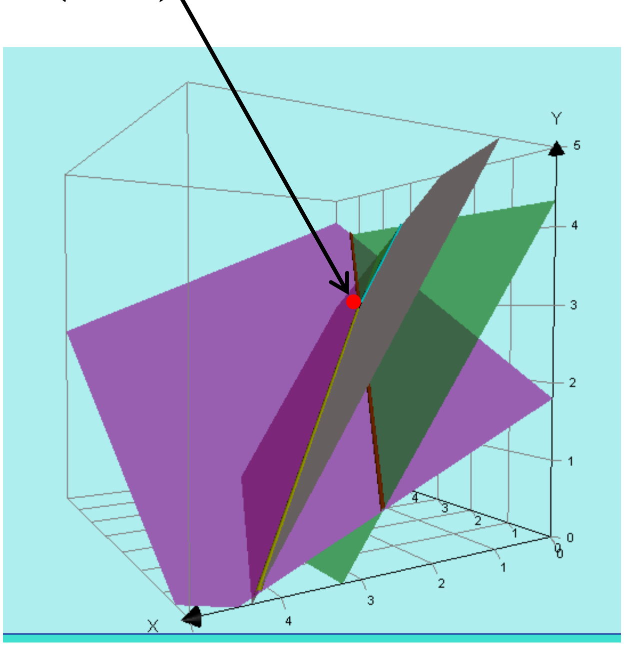
$3y = 9$

$y = 3$

sub in ①: $3 + 6 + z = 13$

$z = 4$

the intersection point is (1, 3, 4)



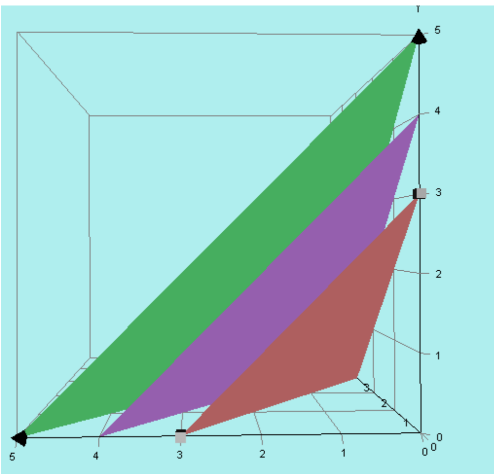
<u>WHEN PLANES GO WRONG!</u>

In 2-Dimensional space, most lines intersect with each other:
eg ***y = 3x – 4 and y = x + 6***
we put 3x – 4 = x + 6 and solve:
2x = 10
x = 5 and y = 11
the lines intersect at (5, 11)

But we find that sometimes lines do not intersect.
This **only** occurs when the lines are **parallel.**

eg ***y = 2x + 6 and y = 2x + 10***
If we try to go through the algebraic process as before, we get equations which have no solution.
2x + 6 = 2x + 10
This leads to no sensible conclusion because if we subtract 2x from both sides we get 6 = 10.
These equations are called ***inconsistent.***

Similarly, there are several cases when 3 planes do not intersect.
1. The simplest case is for **<u>3 parallel planes</u>**.
eg ***x + y + z = 3*** ①
x + y + z = 4 ②
x + y + z = 5 ③



There is no common solution.

As above, the algebra gives us impossible situations.
If we subtract equation ① from ② we get 0 = 1

These equations are also called ***INCONSISTENT***.

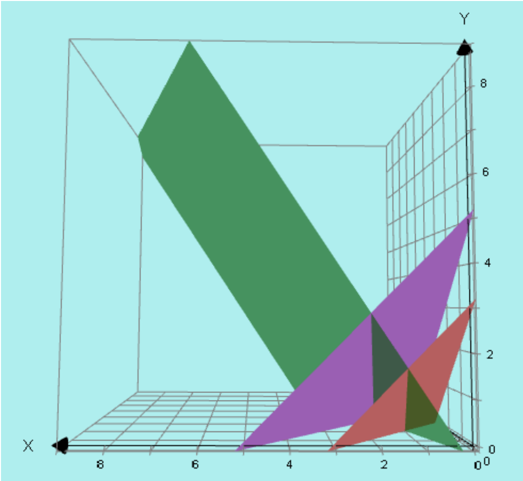
Parallel planes are easy to pick out
eg ***2x + 3y – 4z = 5*** ①
2x + 3y – 4z = 11 ②
4x + 6y – 8z = 13 ③
Notice ③ is just multiplied by 2.
It could have been written as :
2x + 3y – 4z = 6.5

Obviously, equations of parallel plane have the **SAME COEFFICIENTS**.

2.(a) If 2 of the planes are parallel and the 3rd plane crosses them both, there is no intersection of all 3 planes.

eg $x + y + z = 3$ ①
$x + y + z = 5$ ②
$3x - 2y - z = 1$ ③

Again it is easy to pick out that ① and ② are parallel because the coefficients of ***x, y*** and ***z*** are the same.

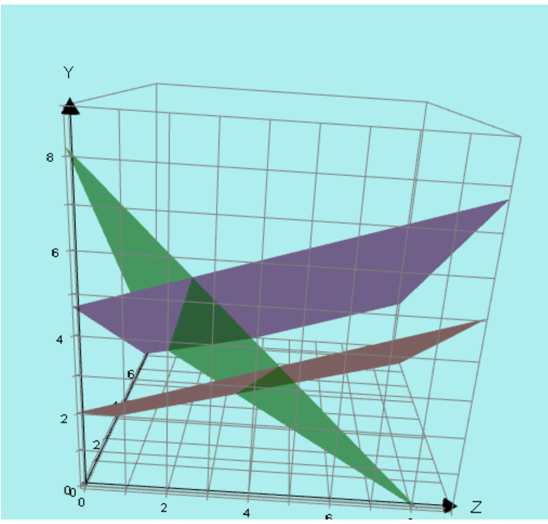


If we **"attempt"** the usual technique for trying to find a unique solution we get impossible situations again.
eg adding ① and③ $4x - y = 4$
and adding ② and ③ $4x - y = 6$
These // line graphs are the projections on the x,y plane of the actual lines of intersection of the planes ① & ③ and ② & ③ respectively.
These equations have no solution and are also called ***INCONSISTENT.***

2(b) Consider these 3 planes:

$2x + 3y - z = 6$ ①
$2x + 3y - z = 14$ ②
$x + y + z = 8$ ③

Clearly planes ① and ② are parallel because they have the **same coefficients** of ***x, y*** and ***z.***
The plane ③ would cross them both as shown below:



Again it is easy to pick out that ① and ② are parallel because the coefficients of ***x, y*** and ***z*** are the same.

If we **"attempt"** the usual technique for trying to find a unique solution we get impossible situations again.
eg adding ① and③ $3x + 4y = 14$
and adding ② and ③ $3x + 4y = 22$
These // line graphs are the projections on the x,y plane of the actual lines of intersection of the planes ① & ③ and ② & ③ respectively.
These equations have no solution and are also called ***INCONSISTENT.***

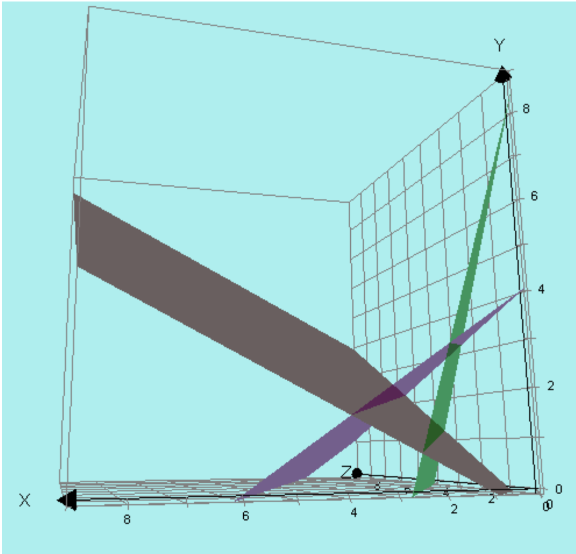
3. (A very odd situation):

Consider these equations:

$x - 2y + z = 1$ ①

$2x + 3y + z = 12$ ②

$3x + y + 2z = 8$ ③



Attempting a solution in the normal way:

Elim z from ① and ②:

② - ①: $x + 5y = 11$ ④

Elim z from ① and ③

2×①: $2x - 4y + 2z = 2$ ⑤

$3x + y + 2z = 8$ ③

③ - ⑤: $x + 5y = 6$ ⑥

Elim z from ② and ③

2×②: $4x + 6y + 2z = 24$ ⑦

③ $3x + y + 2z = 8$

④ - ⑦: $x + 5y = 16$ ⑧

Note: Remember, equations ④, ⑥ and ⑧ are best considered to be 3 parallel lines in the *x, y* plane which are the projections of the actual lines of intersection of the 3 planes.

Clearly the equations ④, ⑥ and ⑧ have the same coefficients of ***x*** and ***y*** but different constant terms.

Looking at the picture, we see that the 3 planes do not intersect in a single point but they intersect forming a **PRISM** shape.

GENERALLY: This always occurs if a linear combination **of 2 of the equations** produces an equation with the **same coefficients** of ***x, y*** and ***z*** as the 3rd equation but with a different **constant term**.

$x - 2y + z = 1$ ①

$2x + 3y + z = 12$ ②

adding $3x + y + 2z = 13$ ***i.e. same coefficients as ③ but constant = 13 not 8.***

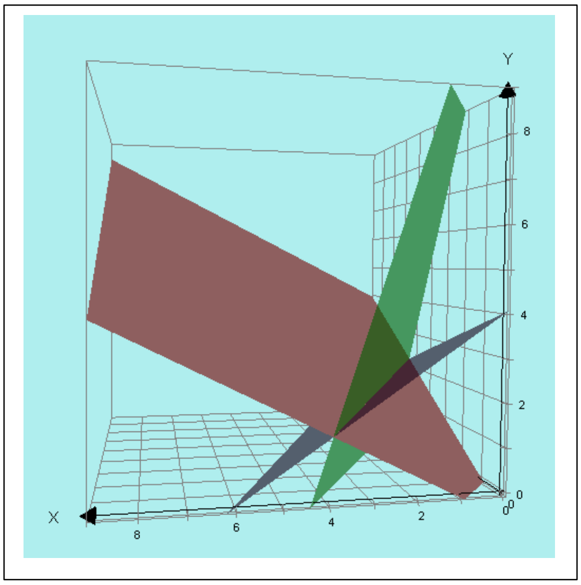
These equations have no solution and are also called ***INCONSISTENT.***

4. (Another very odd situation): Consider these equations:

$x - 2y + z = 1$ ①

$2x + 3y + z = 12$ ②

$3x + y + 2z = 13$ ③



Attempting a solution in the normal way:

Elim ***z*** from ① and ②

② - ①: $x + 5y = 11$ ④

Elim ***z*** from ① and ③

2×①: $2x - 4y + 2z = 2$ ⑤

③ $3x + y + 2z = 13$

③ - ⑤: $x + 5y = 11$ ⑥

Elim ***z*** from ② and ③

2×②: $4x + 6y + 2z = 24$ ⑦

③ $3x + y + 2z = 13$

⑦ - ③: $x + 5y = 11$ ⑧

Clearly the equations ④, ⑥ and ⑧ are the same line, projected on the ***x, y*** plane. So that in this case the 3 planes must also intersect in a single **LINE.**

<u>This means every point ON the line is a solution of the 3 equations.</u>
<u>There are infinitely many solutions.</u>
These equations are called *DEPENDENT*.
GENERALLY, this happens if a linear combination of 2 of the equations EQUALS the 3rd equation.
In this case Equ① + Equ② = Equ③

<u>SPECIAL NOTE:</u> ***Considering $x + 5y = 11$***

If $y = 1$ then $x = 6$ and subs in any of ***①, ② or ③ we get $z = -3$***
so (6, 1, - 3) is one solution.

If $y = 2$ then $x = 1$ and subs in any of ***①, ② or ③ we get $z = 4$***
so (1, 2, 4) is another solution.

If $y = 0$ then $x = 11$ and subs in any of ***①, ② or ③ we get $z = -10$***
so (11, 0, -10) is another solution.

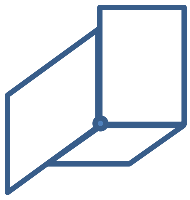
If $y = 3$ then $x = -4$ and subs in any of ***①, ② or ③ we get $z = 11$***
so (-4, 3, 11) is another solution.

The actual equation for the line of intersection of the three planes is:
$x = 1 + 5t, y = 2 - t, z = 4 - 7t$

SUMMARY: GEOMETRICAL INTERPRETATION.

1. If the equations are clearly **INDEPENDENT**, meaning we could not combine two of the equations and produce the 3rd, then there will be a **UNIQUE SOLUTION** where the planes cross at a single point ***(x, y, z)***

eg $x + y + z = 3$
$2x + 3y + 4z = 9$
$3x + 5y + 3z = 11$



2. If the equations all have the same coefficients of ***x, y*** and ***z*** but different constant terms, then they are **three parallel planes** and there is no intersection.
There are no common points of intersection which means there are no solutions.

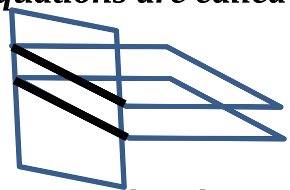
(These equations are called inconsistent.)

eg $2x + 3y + 6z = 5$
$2x + 3y + 6z = 9$
$2x + 3y + 6z = 12$

3. If TWO of the planes can be shown to have the same coefficients of ***x, y*** and ***z*** but different constant terms, then the **two planes are parallel** and the third plane cuts each of them in two parallel lines.
There are no common points of intersection which means there are no solutions.

(These equations are called inconsistent.)

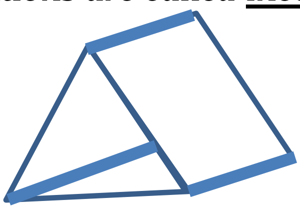
eg $2x + 3y + 6z = 5$
$2x + 3y + 6z = 9$
$5x + 7y + 2z = 12$



4. If a linear combination of two of the equations has the same coefficients of ***x, y*** and ***z*** as the third equation but the constant term is different, the three planes form a triangular PRISM.
There are no common points of intersection which means there are no solutions.

(These equations are called inconsistent.)

eg $2x + 3y + 6z = 5$
$4x + 7y + 2z = 9$
$6x + 10y + 8z = 4$



5. If a linear combination of two of the equations equals the third equation then the three planes intersect in the same line and **there are infinitely many solutions.(Each point on the line is a solution)**

(These equations are called dependent.)

eg $2x + 3y + 6z = 5$
$4x + 7y + 2z = 9$
$6x + 10y + 8z = 14$